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Exploring students' understanding layers in solving arithmagon problems

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ABSTRACT

Previous studies have reported that students' understanding is strongly influenced by their previous knowledge. Specifically, "don't need" boundaries are students' activities with ideas that are no longer bound to previous forms of understanding. Anchored by the importance of knowing such boundaries and students' understanding layers, the present study was designed to explore the characteristics of "don't need" boundaries of students in solving arithmagon problems using a qualitative descriptive approach. To collect the data, test and unstructured interviews were carried out. The test questions were administrated to 23 participants. They were recruited based on indicators of students' written communication skills which appeared in the activity of solving the test questions. To triangulate the data, unstructured interviews were done. Findings suggest that S1 crossed the first "don't need" boundaries when solving the addition and multiplication arithmagon problems. S2 crossed the first and second "don't need" boundaries when solving the addition arithmagon problem and only crossed the first 'don't need' boundaries in solving multiplication arithmagon problems. However, S3 crossed the second "don't need" boundaries in solving arithmagon problem of addition and multiplication. Based on these findings, future research is encouraged to find and explore the third "don't need'" boundaries in solving mathematical problems.

Keywords: "don't need" boundaries, problem-solving, arithmagons

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INTRODUCTION

Mathematical understanding is a basis for thinking about solving mathematical problems in real life. In addition, one of the objectives of the mathematics education curriculum is the development of mathematical understanding abilities. The mathematical understanding strongly supports other mathematical abilities such as communication, reasoning, connection, representation, and problem solving (Lambertus, 2016). Pirie and Kieren theory defines understanding as a process of growth movement that is complete, layered, non-linear, continuous, and non-stop (Gülkılık, Uğurlu, & Yürük, 2015). According to Utami, Sa'dijah, Subanji, and Irawati (2018), six levels of students' mental models in understanding integer concepts are pre-initial mental models, initial mental models, and transition 1, synthetic mental model, transition 2, and formal mental model. Furthermore, Utami, Sa'dijah, Subanji, and Irawati (2019) state that first-grade students at the level of pre-initial mental model do not understand the concept of relationship well; they cannot solve relationship problems that are not functional. Thus, understanding mathematics is a very important part of the process of learning especially learning mathematics.

A strong understanding has been deemed essential for students. The Pirie-Kieren theory of understanding acts as a powerful lens to observe the growth of understanding in a learning activity (Gokalp & Bulut, 2018). It aims to develop students to have the ability to: (1) understand, explain, and apply mathematical concepts and relationships between concepts accurately, effectively, efficiently, and precisely in solving problems; (2) use reasoning, generalize, and gather evidence about mathematical statements; (3) solving problems according to the stages of problem-solving; (4) communicating ideas in the form of mathematical models; and (5) confident in solving problems. According to Sengul and Argat (2015), most of the time, students have difficulties understanding the information presented to them due to a lack of prior knowledge. The learning process in this era of globalization is still not going well. This is due to the lack of interaction between teachers and students, and between students and other students. Even the test results of most students indicate the lack of basic understanding skills they should have learned in elementary school and in everyday application problems (Pramudiani, Zulkardi, Hartono, & Amerom, 2011). Students' knowledge is strongly influenced by their previous knowledge (Aziz, Supiat, & Soenarto, 2019). Previous knowledge is likely to be considered as the absolute truth, which could be applied in various contexts.

Generally, students will have knowledge and understanding of mathematical concepts if they can (1) explain concepts verbally and in writing; (2) analyze and make examples and not examples; (3) present concepts using mathematical models; (4) represent them in other forms; (5) identify various meanings and interpretations of concepts; (6) analyze the properties and understand the terms which define a concept; and (7) compare them (Sumartini & Priatna, 2018). Furthermore, mathematics learning aims at achieving meaningful understanding and must lead to developing the ability to connect mathematics with different ideas, understanding the relationship of different ideas so that the students can construct a comprehensive understanding and apply mathematics in other contexts. A good understanding of mathematics starts from the previous understanding possessed by the students. Therefore, the students' understanding greatly affects the quality of their understanding at the next level of education.

Pirie and Kirien's mathematical understanding levels include primitive knowing, image-making, image having, property noticing, formalising, observing, structuring, and inventising (Wright, 2014). The understanding layer of each concept is dynamic and always changes from one layer to another (Sa'dijah, Rahayuningsih, Sukoriyanto, Qohar, & Pujarama, 2021). When a person wants to progress to the next understanding layer, they often return to the previous layer of understanding (Fauziyyah & Kriswandani, 2018).

Some important features in Pirie-Kieren's theory are folding back, "don't need" boundaries, complementary aspects of acting and expressing, and intervention (Nakamura & Koyama, 2018). Folding back is a very important activity for understanding, which reveals the un-directional nature of mathematics understanding. This happens when the students face problems in any layer which cannot be resolved immediately, so they need to retreat to deeper layers to broaden their understanding which is inadequate at that time (Sengul & Argat, 2015). "don't need" boundaries is the students' activity with ideas which are no longer clearly bound to previous forms of understanding but rather they are embedded in a new level of understanding and are easily accessible if needed (George, 2017). Complementary aspects of *acting* and *expressing* are activities experienced at the level between primitive knowing and inventing as acting and expressing. *Acting* is a mental or physical activity that encompasses all previous understandings, provides continuity with deeper levels, and expresses activities that are generally verbal statements that give different substances from a certain level. *Reflection* is often understood as a component of the act of *acting* because it is an activity in combining the process of how to build the previous understanding. On the other hand, in *expressing*, it is necessary to review and interpret the components involved in *acting*. *Intervention* is an action that stimulates both internally and externally that directs students to review their current understanding (Gülkılık et al., 2015).

Previous studies on understanding using Pirie and Kieren's theory have been carried out by Mabotja, Chuene, Maoto, and Kibirige (2018), Fauziyyah and Kriswandani (2018), and Gokalp and Bulut (2018). Empirically, Mabotja et al. (2018) argue that learners' effective *folding back* is a powerful tool to enhance their geometric reasoning. In the same vein, Fauziyyah and Kriswandani (2018) described the profile of understanding layers of the cone section concept, and Gokalp and Bulut (2018) revealed that there was a relationship between the students' preference on the use of different types of representations and attained an understanding level of multiplication of fractions.

A salient feature in Pirie-Kieren's theory of understanding is "don't need" boundaries. According to Pirie-Kieren's theory, one of the strengths of mathematics is the ability to operate at a symbolic level without referring to the basic concepts or images which is shown in the thicker lines in Figure 1. Those lines show an increase in abstract understanding (Borgen, 2006) and separate the model into four parts. "Don't need" boundaries means that the students no longer need specific actions that have been taken in the layers within the boundaries and they can work with layers of understanding which are more general and abstract beyond the boundaries (Nakamura & Koyama, 2018). The first "don't need" boundaries occur between image making and image having. When a person has a picture of a mathematical idea, they do not need actions or specific examples of image making. The second "don't need" boundaries occur between the noticing properties and formalising. A person who has a formal mathematical idea does not need a picture. Similar to the relationship between image having and property noticing, it involves observing, by definition, focusing on current formalising. The third "don't need" boundaries occur between observing and structuring. A person with a mathematical structure does not need the meaning brought to them at any deeper level.

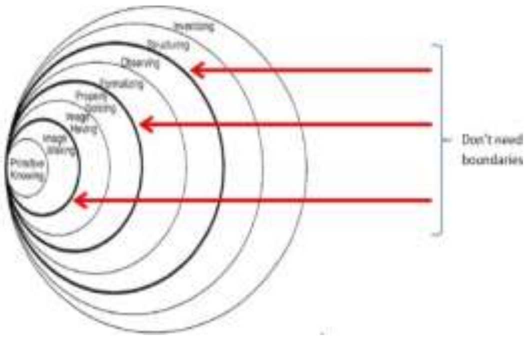


Figure 1. Pirie-Kieren's Model Showing "Don't Need" Boundaries (Borgen, 2006)

In a recent study, Junsay (2016) states that conceptual understanding is significantly related to problem-solving abilities while solving challenging problems for students with an adequate knowledge base do not only encourage conceptual understanding but also improve cognitive development (Retnowati, Fathoni, & Chen, 2018; Purnomo, Sa'dijah, Cahyowati, Nurhakiki, Anwar, Hidayanto, & Sisworo, 2021; Subanji, Nusantara, Rahmatina, & Purnomo, 2021). Considering the importance of conceptual understanding in solving problems, further research is needed on understanding students' concepts in-depth and continuously. Afriyani, Sa'dijah, Subanji, and Muksar (2018) in their research also revealed that there is still an opportunity to conduct further research in examining the characteristics of students' expanded mathematical understanding at the abstract level. Thus, this study intends to explore the "don't need" boundaries of students in solving arithmagon problems.

Arithmagon is a specific number triangle. In an arithmagon, there is a number in each corner of the triangle and the total is between the angle numbers (the sides of the triangle) (Mason, Burton, & Stacey, 2010). An arithmagon problem is a non-standard problem that requires new thinking. The students must think backward to find how and what numbers should be placed in the arithmagon. The main idea is to understand that if the numbers in the middle of the sides are the same, then the numbers in the lower corner must be the same. When students write the solution method, they must reflect back on what they have done and why that particular method works. By so doing, they practice explaining their own thoughts. In the case of an arithmagon problem, for example, the main idea is usually associated with realizing that the number of parties plays an important role in determining whether the required numbers can be uniquely determined or not. However, for most of the students, this realization is not surprising, but after working with various cases, it produces various results; that is, after the process of knowledge generation (Guajardo, 2004). As a flexible context for producing problems with various difficulties, an arithmagon presents a good opportunity for school-age students in order to build algebraic reasoning skills while analyzing the nature of numbers and their operations (Burke, Kehle, Kennedy, & John, 2006).

In relation to arithmagons, previous studies have examined it extensively (see Liang, 2003); Lin, Kuo, & Yang, 2014; Laine, Ahtee, Näveri, Pehkonen, & Hannula 2018). Liang (2003) used arithmagon questions to explore the problem-solving abilities of prospective Singapore mathematics teachers. Lin, Kuo, and Yang (2014) used the arithmagon problems to compare prospective American and Taiwanese teachers in solving the problem of triangular arithmagons. Laine, Ahtee, Näveri, Pehkonen, and Hannula (2018) used arithmagon questions to determine the teacher's influence on the quality of students' written explanations in solving nonstandard problems. In the present study, the arithmagons questions are used to explore students' layers of understanding in solving arithmagon problems.

Exploring layers of understanding in solving arithmagon problems of junior high school students helps reveal to what extent is the students' layers of understanding. For this reason, this study is focused on describing students' unnecessary boundaries ("don't need" boundaries) in solving arithmagon problems. Therefore, the study seeks to construe the "don't need" boundaries of junior high school students in solving arithmagon problems.

METHOD

The present study employed a descriptive qualitative approach to data gathering. In order to explore junior high students' layers of understanding in solving arithmagon problems, the Piere-Kieren's theory which consists of eight layers of understanding was used in this study. In particular, the study focused on the boundaries that the junior high school students do not need in solving arithmagon problems. The subjects were selected based on indicators of students' written communication skills which appeared in the activity of solving the test questions. The arithmagon problem questions were given to 23 junior high school students of the second grade in Malang regency. The indicators are summarized in Table 1.

Table 1. Indicators of Students' Written Communication Ability in Solving Test Questions

Characteristic	Indicator
The existence of diagrams/drawings/ mathematical modeling that is appropriate	The students can make diagrams/drawings/ mathematical modeling that is appropriate according to the information presented in the problem
Using effective sentences	The students can use effective and clear sentences in solving problems
Structured solution strategy	The students can solve the problem in a structured manner

To collect the data, tests and unstructured interviews were conducted. The test consists of triangular arithmagon addition and multiplication questions. The problems were created to obtain the characteristics of the understanding layer focusing on the "don't need" boundaries feature. Furthermore, the unstructured interviews were conducted to dig up information based on the results of each subject's work. Hence, the questions in the interviews were developed without guidelines, depending on each subject's answer. The test instrument is shown in Figure 2.

Solve the following questions thoroughly!

Arithmagons are polygons with specific numbers on the points (circles) that determine numbers on the sides (squares).

In addition arithmagon, numbers on the sides (box) are the sum of the two numbers on the point (circle) that lie between the sides.

1. In the Figure on the right, the number in each box is equal to the number of numbers in the two circles adjacent to the box. Determine the exact number to fill the circle in the Figure on the side! Explain your answer!

In multiplication arithmagons, the numbers on the sides (squares) are the product of the two numbers on the point (circle) that lie between these sides.

2. In the Figure on the right, the numbers in each box are the same as the product of the numbers in the two circles adjacent to the box. Determine the exact number to fill the circle in the Figure on the side! Explain your answer!

Figure 2. Test Instrument

The indicators of understanding layer in solving arithmagon problems are shown in Table 2.

Table 2. Indicators of Understanding Layer in Solving Arithmagon Problems

Understanding Layer	Indicator of Understanding Layer
Primitive knowing	1. The students possess the concept understanding of triangular arithmagon in which the students analyse the relation between the triangle angles and the triangle sides. 2. The students possess the understanding of the concept of triangular arithmagon with the understanding possessed by them which is arithmetic (integer operation). 3. The students understand the concept of triangular addition arithmagon in which 2 numbers put on the vertex of the triangle that are close to one another if added will produce a number listed on the side of the triangle. 4. The students understand the concept of triangular multiplication arithmagon in which 2 numbers placed at the vertex of a triangle that is close to each other if multiplied will produce a number listed on the side of the triangle.
Image making	1. The students provide several symbols that will be sought using variables (letters). 2. The students explain the concept of triangular addition arithmagon as a whole that the students should determine 3 integers placed at every angle of the triangle in which 2 close integers if added will produce the result listed on the side of the triangle formed. 3. The students explain the concept of triangular multiplication arithmagon as a whole that the students should determine 3 integers placed at every angle of the triangle in which 2 close integers if multiplied will produce the result listed on the side of the triangle formed. 4. The students determine various integer alternatives using trial and error. 5. The students determine various integer alternatives mentally.
Image having	1. The students associate the triangular addition arithmagon concept with the linear equation system. 2. The students associate the triangular multiplication concept arithmagon with the concept of exponent.

	3. The students can rule the linear equation in the linear equation system. 4. The students can rule the formulated linear equation system solution. 5. The students can rule the solution of exponent formulated.
Property noticing	1. The students are capable of determining the relation of 3 numbers in the vertex and 3 numbers in the side of the triangle. 2. The students analyze the pattern in the triangular addition arithmagon. 3. The students analyze the pattern in the triangular multiplication arithmagon.
Formalising	1. The students are capable of stating the concept of triangular addition arithmagon abstractly according the existing traits. 2. The students are capable of stating the concept of triangular multiplication arithmagon abstractly according to the existing traits.
Observing	1. The students are able to use the formula from triangular addition arithmagon in order to solve the problem of triangular addition arithmagon on the decimal numbers and multi-digit integers. 2. The students are able to use the formula from triangular multiplication arithmagon in order to solve the problem of triangular multiplication arithmagon on the decimal numbers and multi-digit integers.
Structuring	The students are able to analyze the relationship between triangular addition arithmagon and triangular multiplication arithmagon.
Inventising	1. The students are able to provide further question related to triangular arithmagon. 2. The students can solve further problem questions made. 3. The students can explore the triangular arithmagon problem questions on fractions or the pentagon arithmagon, square arithmagon, and so on.

The data collection procedure began with the selection of subjects based on predetermined written communication skill indicators. The result of the selected subjects' answers was analyzed based upon the understanding layers from Pirie Kieren's theory. The subject's results were then further explored based on the "don't need" boundaries in solving arithmagon problems. In order to determine the accuracy of the data, triangulation method was done by exploring data obtained from tests and interviews. The "don't need" boundaries indicators were modified from Thom and Pirie's (2006) work. The "don't need" boundaries indicators in solving arithmagon problems are shown in Table 3.

Table 3. The Indicators of *"Don't Need" Boundaries* in Solving Arithmagon Problems

"Don't Need" Boundaries	Description	Indicator
First "don't need" boundaries	• Occurs somewhere between image making and image having • Occurs when someone has a picture about mathematical idea and does not need actions or specific examples from **image** making • Focuses on image having	1. The subjects are able to determine the linear equation system verbally without making models and its symbols. 2. The subjects are able to solve addition and multiplication arithmagon without explaining the addition and multiplication arithmagon concept.
Second "Don't Need" Boundaries	• Occurs between property noticing and formalising	The subjects are able to construct a triangular

	• Occurs when someone has a formal mathematical idea and does not need the picture • Focuses on formalising	addition and multiplication arithmagon formula according to the existing traits without analyzing the pattern in the triangular addition and multiplication arithmagon.
Third " Don't Need" Boundaries	• Occurs between observing and structuring • Occurs when someone with mathematical structure and does not need meaning brought to them at any deeper level • Focuses on structuring	The subjects are able to relate among the vertex and the side of a triangular arithmagon either the addition or multiplication without using formula.

RESULTS AND DISCUSSION

Results

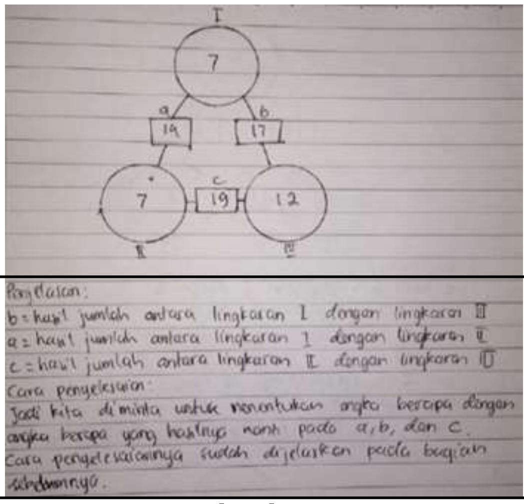
The work result of 23 junior high school students showcases some similarities in terms of the process of solution. Those similarities were classified into 3 categories. There were 6 students in the first category, 11 students in the second category, and 6 students in the third category. Furthermore, 1 student was selected from each category as the research subject. Subject 1 represented category 1 and was referred to as S1, Subject 2 as S2 and represented category 2, while Subject 3 as S3 which represented category 3. The subjects' categories are illustrated in Table 4.

Table 4. Subject Categories

Subject	Indicators that appear
S1	The students can make mathematical diagrams/ drawings/ modelings precisely according to the information presented in the problem using effective and clear sentences in solving problems but cannot solve the problem in a structured manner.
S2	The students can make mathematical diagrams/ drawings/ modelings precisely according to the information presented in the problem but the subjects do not use effective and clear sentences in solving problems and cannot solve the problem in a structured manner.
S3	The students can make mathematical diagrams/ drawings/ modelings precisely according to the information presented in the problem using effective and clear sentences in solving problems in a structured manner.

Subject 1 (S1)

S1's work result on question number 1 is illustrated in Figure 3.



Translated Version

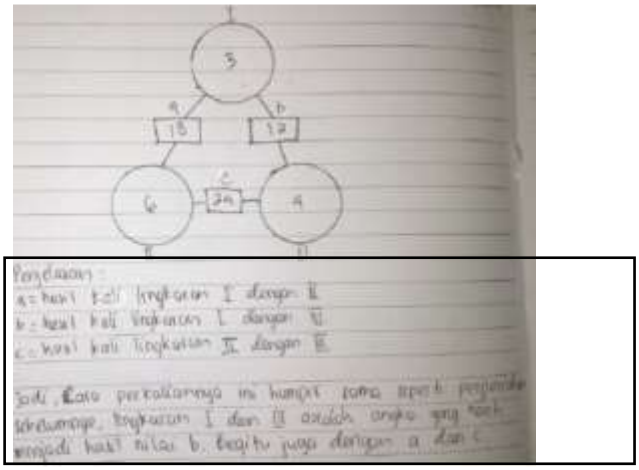
Explanation:
b: the result of the sum between circle I and circle III
a: the result of the sum between circle I and circle II
c: the result of the sum between circle II and circle III

Figure 3. S1's Work Result of Addition Arithmagon Problems

Figure 3 indicates that S1 understands the concept of triangular arithmagon as the subject analyzes the relationship between the vertex of and the sides of a triangle. S1 also understands the concept of triangular arithmagon with the previously possessed understanding that is arithmetic (integer operation). S1's explanation also shows that S1 understands the concept of triangular addition arithmagon in which *w* numbers are placed at the vertex of adjacent triangles and if added together it produces the magnitude of the number listed on the sides of the triangle. S1 understands the overall concept of triangular arithmagon which is to determine 3 integers placed at each vertex of the triangle in which 2 adjacent integers if added together produce a result listed from the sides of the formed triangle. This indicates that S1 is in the primitive knowing understanding layer. Based on the interview with S1, it turns out that before giving an explanation of the answer, S1 had guessed the answer even before it was written on the answer sheet which means S1 is in the image making understanding layer. It is because S1 was able to determine various alternative integers by trial and error. In addition, S1 also provides a symbol of the number sought using variables (roman numerals and a, b, c letters).

S1 could also determine the relationship of 3 numbers at the vertex and on the sides of the triangle in the explanation which, if modeled into a mathematical form such as $\mathrm{a} = \mathrm{I} + \mathrm{II}$, $\mathrm{b} = \mathrm{I} + \mathrm{III}$, and $\mathrm{c} = \mathrm{II} + \mathrm{III}$. This shows that S1 is in the property noticing understanding layer. It turns out that S1's answer on the answer sheet is incorrect because S1 answered that the result of III is 12 while the answer should be. S1 was aware of this when the interview was conducted and re-checked the answer by calculating the answer using the subject's fingers that $17 \neq 7 + 12$, it indicates that S1 is in the image having understanding layer since the subject can solve the linear equation system constructed using the explanation. Thus, S1 returns to the image having understanding layer from the property noticing layer. Pirie and Kieren identified this return process as something important for the development which is called folding back.

S1's Work Result of Multiplication Arithmagon Problems is shown in Figure 4.



Translated Version

Explanation:

a: product of circle I by circle II

b: product of circle I by circle III

c: product of circle II by circle III

Figure 4. S1's Work Result of Multiplication Arithmagon Problems

Figure 4 shows that S1 understands the concept of triangular multiplication arithmagon in which 2 numbers placed at the vertex of adjacent triangles when multiplied produce the magnitude of the numbers listed on the sides of the triangle. It signifies that S1 is at the primitive knowing understanding layer. S1 also provides the symbol of the number sought with variables (roman numerals). In addition, S1 also understands the concept of a triangular multiplication arithmagon as a whole that is having to determine 3 integers placed at each vertex of the triangle where two adjacent integers if multiplied will obtain a result that is listed on the sides of the triangle formed. The results of the interview with S1 show that before giving an explanation of the answer, S1 had guessed the answer before it was written on the answer sheet. The result of the work and interview indicates that S1 is in the image making understanding layer.

S1 is also in the image having understanding layer since the subject can relate the concept of triangular multiplication arithmagon to the concept of exponent even though in sentence form. In addition, S1 is able to determine the solution from exponent that is compiled but the subject does not need a mathematical model (image making) in order to solve them. This shows that S1 has a picture of mathematical ideas but does not require action from image making. The explanation from S1 has shown that S1 is in the image having understanding layer.

Subject 2 (S2)

S2's work result of addition arithmagon problems is shown in Figure 5.

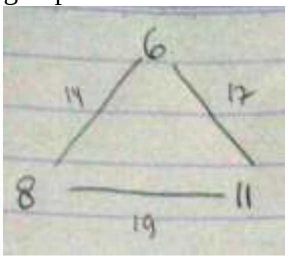


Figure 5. S2's Work Result of Addition Arithmagon Problems

S2 did not provide any explanation for the answer. When interviewed, it turned out that S2 answered the questions only with feelings and immediately wrote them down. S2 determined various alternative answers by trial and error which indicates that S2 is in the primitive knowing understanding layer. When the interview was conducted, S2 explained that before writing on the answer sheet, the subject only counted mentally and immediately re-drew the triangular arithmagon on the answer sheet. It indicates that S2 is in the image making understanding layer. Based on the explanation during the interview, S2 answered the question based on the definition of a triangular addition arithmagon and could explain the relationship of 3 numbers at the vertex and 3 numbers on the side of the triangle. Furthermore, S2 could also explain the patterns which exist in the triangular addition arithmagon in order to answer the problem. Therefore, S2 is in the property noticing understanding layer. It is because S2 solve the problem correctly based on the strategy chosen so that S2 has experienced folding back as it has occurred the process of returning to the image having understanding layer from property noticing layer.

S2 can solve the problem of addition arithmagon correctly (image having) without having to calculate or write down the numbers and symbols (image making). Results of the interview with S2 show that the subject can state the concept of triangular addition arithmagon abstractly based on the existing traits (formalising) without analyzing the existing pattern in triangular addition arithmagon (property noticing).

The result of S2's work is seen in Figure 6.

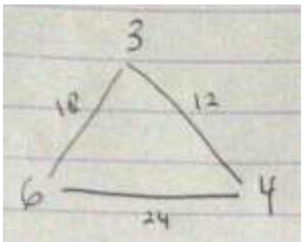


Figure 6. S2's Work Result of Addition Arithmagon Problems

Similar to the answer for question number 1, S2 did not provide the reason in detail. From the result of S2's answer, it is seen that S2 understands the concept of triangular arithmagon with previously possessed understanding that is arithmetic concept (integer operations), which means that S2 is in the primitive knowing understanding layer.

In the previous interview result, it is shown that S2 has linked the concept of triangular multiplication arithmagon concept to the concept of exponents. It means that S2 is in the image having understanding layer. In the explanation, S2 solves question number 2 according to the definition of triangular multiplication arithmagon on the question that is the result of the multiplication between the angular numbers is the side of the triangle. It shows the fact that S2 has analyzed the existing pattern in triangular multiplication arithmagon in order to solve the problem. Thus, S2 is in the property noticing understanding layer.

Subject 3 (S3)

S3's Work Result of Addition Arithmagon Problems is shown in Figure 7.

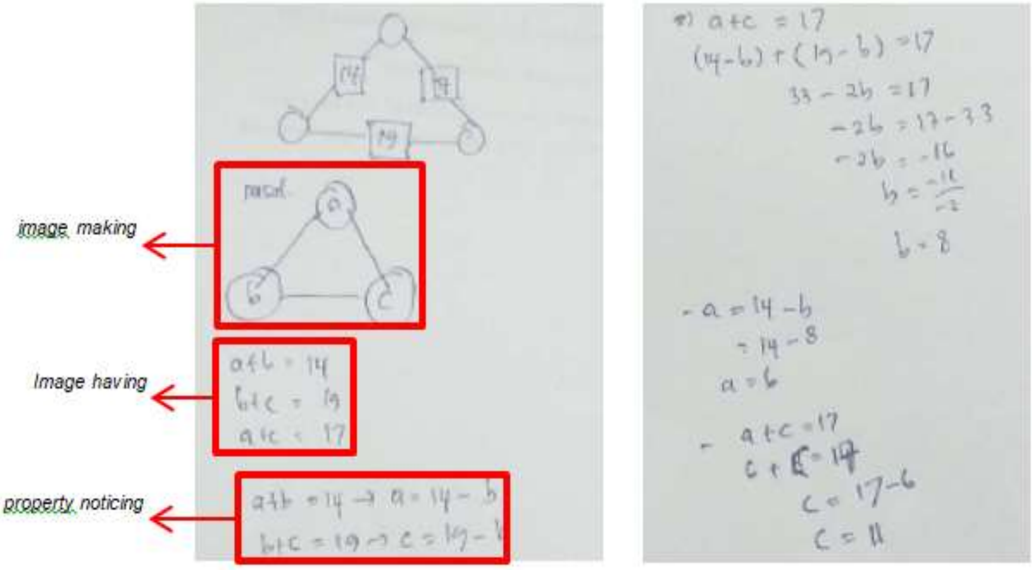


Figure 7. S3's Work Result of Addition Arithmagon Problems

S3 understand the concept of triangular addition arithmagon in which S3 is capable of analyzing the relation between the vertex and the sides of a triangle. S3 also understands the concept of triangular arithmagon with the previously possessed understanding that is arithmetic (integer operations). This means that S3 is in the Primitive Knowing understanding layer. The subject connected the concept of triangular addition arithmagon to the concept of linear equation system by providing the symbol of the number sought using variables (letters a, b, and c). Therefore, it indicates that S3 is in the image making understanding layer. S3 is in the image having understanding layer as well since the subject can determine the solution of the linear equation system constructed. S3 is able to determine the relation of 3 numbers at the vertex and 3 numbers which are on the side of the triangular addition arithmagon. This means that S3 is in the property noticing understanding layer. Through the interview, S3 can state the concept of triangular addition arithmagon abstractly based on the existing traits. This indicates that S3 is in the formalising understanding layer.

S3's Work Result of Multiplication Arithmagon Problems is shown in Figure 8.

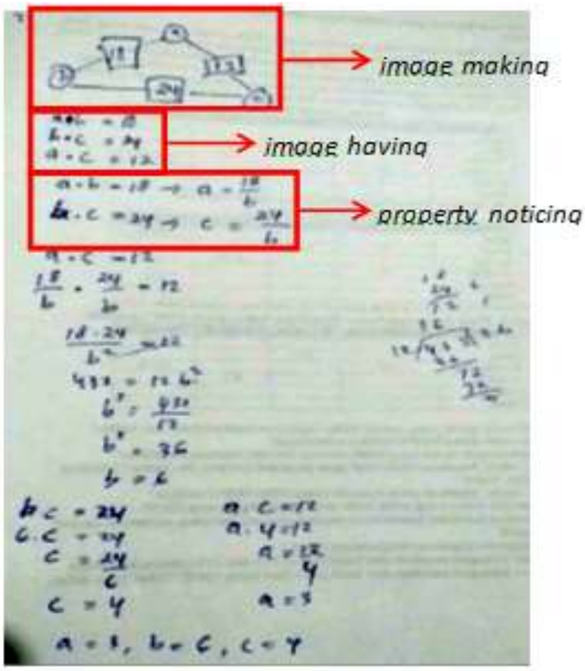


Figure 8. S3's Work Result of Multiplication Arithmagon Problems

Figure 8 shows that S3 understands the concept of triangular multiplication arithmagon in which two numbers placed at the vertex of adjacent triangles, when multiplied, will produce a magnitude of numbers listed on the sides of the triangle. It means that S3 is in the primitive knowing understanding layer. S3 also provides symbols of the number sought with variables (letters) and when interviewed S3 was able to explain the concept of triangular arithmagon as a whole in which the subject should determine 3 integers placed at each vertex of the triangle where the 2 adjacent integers if multiplied will result in the numbers from the sides of the triangle formed. It indicates that S3 is in the image making understanding layer. S3 can also associate the concept of triangular multiplication arithmagon to the concept of exponents and determine the solution from the exponent made. It shows that S3 is in the image having understanding layer. From Figure 8, S3 is seen to be able to determine the relation of 3 numbers at the vertex and 3 numbers on the side of the triangle. Therefore, it can be concluded that S3 is in the property noticing understanding layer. Based on the interview conducted, S3 could abstractly explain the concept of triangular multiplication arithmagon according to existing traits. This indicates that S3 is in the formalising understanding layer. The excerpt of the interview indicating that S3 is also in the formalising understanding layer s shown below.

DISCUSSION

The present study attempts to explore the "don't need" boundaries of junior high school students in solving arithmagon problems. Based on the analysis, S1 had a picture of a mathematical idea, but the subject did not require action or specific examples of image making. In this case, "don't need" boundaries have occurred because S1 no longer required specific action carried out in layers within the boundary (image making). However, S1 could solve the addition arithmagon problem with the understanding layer outside of the boundary (image having). Image having occurred just outside of the first "don't need" Boundaries in the Pirie-Kieren Model because of not relying on the more specific inner understanding (Guner & Uygun, 2019). S1 has crossed the first "don't need" boundaries since in order to deliver the idea, which is far off the limit, it does not require deeper understanding (image making) which gives rise to external knowledge (image having).

S2 has experienced folding back because there has been a process of returning to the image having an understanding layer from the property noticing layer. According to Pirie-Kieren's theory, when a person is in the outer layer of understanding, in this case, property noticing, which is then faced with a new problem, it is necessary to return to the deeper understanding layer to study and modify the ideas present with a thicker understanding of a concept, this process is understand as 'folding back' (Martin & Towers, 2016). S2 has crossed the first “don’t need” boundaries because, according to Pirie-Kieren’s theory that students who work outside the “don’t need” boundaries do not need to understand certain inner layers to come up with the external knowledge (Thom & Pirie, 2006; George, 2017). In addition, S2 can also be considered to think intuitively because the subject answered the questions spontaneously. It means that the students understand information spontaneously, quickly, automatically, and not introspectively (Dehaene, 2009).

S2 passed the second “don’t need” boundaries in solving the addition arithmagon because S2 had a formal mathematical idea (formalising) and did not require a picture of the property noticing. Based on Pirie-Kieren’s theory, formalising occurs precisely outside the second “don’t need” boundaries (Thom & Pirie, 2006; Guner & Uygun, 2019). S2 solved the problem of multiplication arithmagon correctly (image having) without calculating or writing down the numbers and symbols (image making). Thus, S2 has crossed the first “don’t need” boundaries because according to Pirie-Kieren’s theory, the students who work outside the “don’t need” boundaries do not need to understand certain inner layer in order to bring out the external knowledge (Thom & Pirie, 2006; George, 2017)

S3 has crossed the second “don’t need” boundaries in solving the addition arithmagon because the subject had formal mathematical ideas (formalising) and did not require an overview of the property noticing. Formalising occurs just outside of the second “don’t need” boundaries according to the Pirie-Keiren Model (Guner & Uygun, 2019).

S3 has been able to state the concept of triangular multiplication arithmagon abstractly based on the existing traits. Formalizing occurs right outside the second “don’t need” boundaries in Pirie-Kieren’s model and there is no need to analyze the patterns which exist in the triangular multiplication arithmagon. According to Pirie-Kieren’s theory, formalising occurs right outside the second “don’t need” boundaries (Thom & Pirie, 2006; Guner & Uygun, 2019).

The described research result shows that the three subjects were only able to cross the first and second “don’t need” boundaries. It is in line with the opinion from Guner & Uygun (2019) who state in their research that students can pass the first and second “don’t need” boundaries but they cannot advance their understanding of the third “don’t need” boundaries. Gülkılık, Uğurlu, & Yürük (2015) shows that proceeding the second “don’t need” boundaries between the level of property noticing and formalising is not an easy task for the students and it might take time. Furthermore, Gülkılık et al. (2015) state that the reason for not paying attention to the movements outside the third “don’t need” boundaries is possibly due to the lack of students’ experience in the observing, structuring, and inventory level making it difficult for the students to build formal understanding.

Pirie and Kieren state that “don't need” boundaries means the students do not always have to be aware of the deepest understanding layer. The thick ring shows that students’ mathematical understanding beyond the “don't need” boundaries do not require to refer to deeper forms of understanding instead it can be accessed if called for. However, S1 and S2 have a process to a deeper layer (folding back). Folding back is defined as not only as a memory of a mathematical experience or a piece of information but as a provider of means through which students can reconstruct, reintegrate, or re-evaluate the previously possessed knowledge so that it can function in the outer layer with “thicker” understanding (Thom & Pirie, 2006). Folding back occurs because the students need to go back to the deeper layer to solve the problems (Guner & Uygun, 2019). In addition, the folding back movement is needed when the students cannot immediately solve the problem with current understanding. Folding back helps the students broaden their understanding on mathematics (Lawan, 2011).

CONCLUSION

Guided by the Pirie-Kieren’s model, it can be concluded that S1 has passed the first “don’t need” boundaries because the subject could determine and solve addition and multiplication arithmagon problems without explaining the concept of addition and multiplication arithmagon. This means that S1 works outside of the “don’t need” boundaries (image having) but does not require an understanding of the inner layer (image making) in order to come up with the external knowledge.

S2 has passed the first and second "don't need" boundaries when solving the addition arithmagon problem. The subject crossed the first "don't need" boundaries due to the fact that S2 can solve the problem of addition arithmagon correctly (image having) without having to calculate or write down the numbers and symbols (image making). S2 crossed the second "don't need" boundaries when the subject could mention the concept of triangular arithmagon abstractly based on the existing traits (formalising) but it is not necessary to analyze the patterns which exist in triangular addition arithmagon (property noticing). However, in solving multiplication arithmagon, S2 only went through the first "don't need" boundaries because the subject could solve the multiplication arithmagon problem correctly (image having) without having to calculate or write down the numbers and symbols (image making).

S3 has passed the second "don't need" boundaries in solving the addition and multiplication arithmagon because the subject could express mathematical ideas in solving formal arithmetic and multiplication arithmagon. In addition, the subject did not require a specific description of property noticing.

The study's findings implicate future research to find and explore the third "don't need" boundaries in solving mathematical problems.

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